

# Homometric sets and Z-related chords

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## 1 Introduction

**The interval and Patterson functions** For a given chord  $P$ , the *interval function* measures the number of ways that the interval  $x$  can be spanned between members of  $P$ :

$$IFUNC_x(P) = |\{a - b \equiv x, a, b \in P\}|.$$

Example:  $P = \{0, 1, 4, 6\}$  in  $\mathbb{Z}_{12}$ . There are four ways to move by 0 (unison) and 1 way to move by each of the other intervals in 12-tone equal temperament (12-tet). Writing the interval function as a vector where the  $i$ th entry (beginning with  $i = 0$ ) is  $IFUNC_i(P)$ , we have

$$IFUNC(P) = [411111111111].$$

Mathematicians call this the *Patterson function*, which is defined as the convolution of  $P$  with  $P^*$ . Thus

$$IFUNC(P) = Patterson(P) = P * P^*.$$

One of the underlying assumptions of the interval function is that a chord's "quality" is in large part a result of the quality of its constituent intervals.

**Z-related chords and homometric sets** If two chords have the same interval function, musicians say that they are *Z-related*. In the language of Patterson functions, if  $P * P^* = Q * Q^*$ , then  $P$  and  $Q$  are *Z-related*.

Example:  $P = \{0, 1, 4, 6\}$  and  $Q = \{0, 1, 3, 7\}$  are *Z-related*, since their interval functions are identical.

Mathematicians say that these sets are *homometric*. Homometric metric sets were first studied in the 1940s by Bullough and others.

By the convolution theorem, the Fourier transform of  $P * P^*$  is

$$\hat{P} \times \hat{P}^* = |\hat{P}|^2.$$

leading to an equivalent definition.  $P Z Q$  if and only if  $|\hat{P}(z)| = |\hat{Q}(z)|$ . That is,  $P$  and  $Q$  are  $Z$ -related if and only if the moduli of their Fourier transforms are equal.

## 2 The $Z$ -relation problem

**Necessary conditions** An important question is what are the necessary conditions for a set to possess a non-trivial  $Z$ -related/homometric partner. This is referred as the  $Z$ -relation problem.

Example: Let  $P = \{0, \sqrt{2}, \sqrt{2} + 3, 6\}$ . Is  $P$  non-trivially  $Z$ -related to any other set? If so, which one(s)?

### Understanding of $Z$ -relations in Music Theory

1. Sets related by transposition and/or inversion are *trivially*  $Z$ -related.
2. If  $P$  and  $Q$  in  $\mathbb{Z}_n$  are  $Z$ -related, then their complements with respect to  $\mathbb{Z}_n$ ,  $\bar{P}$  and  $\bar{Q}$ , are also  $Z$ -related.
3. In  $\mathbb{Z}_{2n}$ ,  $n$ -note sets are  $Z$ -related to their complements.
4. Non-trivial  $Z$ -relations occur for sets of 4 or more notes.
5. Non-trivial  $Z$ -relations do not occur for equal tempered systems of fewer than 8 pitch-classes.
6. Soderberg's "Q-inversion" preserves interval content.

## 3 Some cases of $Z$ -relations

**Case 1** (Bullough, Rosenblatt)

$$\{0, x, x + 3, 6\} Z \{0, x + 3, 6, x + 6\}$$

Examples:

For  $x = 1$ ,  $\{0, 1, 4, 6\}$  (4-Z15)  $Z$   $\{0, 4, 6, 7\}$  (4-Z29).

For  $x = \sqrt{2}$ ,  $\{0, \sqrt{2}, \sqrt{2} + 3, 6\} Z \{0, \sqrt{2} + 3, 6, \sqrt{2} + 6\}$ .

Noting that the second chord in Case 1 is inversionally related to  $\{-x, 0, -x + 3, 6\}$  leads to an alternate formulation:

$$\{0, x, x + 3, 6\} Z \{-x, 0, -x + 3, 6\}.$$

These two different formulations of the same case can be extended in a separate manner. (See cases 3 and 4 below.)

Case 1 generates all possible  $Z$ -related pairs of 4 notes, except the following case. (This was proved by Rosenblatt.)

**Case 2** In  $\mathbb{Z}_{13}$ ,

$$\{0, 1, 4, 6\} Z \{0, 2, 3, 7\}.$$

**Case 3** We can extend the alternate formulation of Case 1 in the following manner. Replace the minor third by two inversionally symmetric sets where the axes of symmetry are a minor third apart.

Let  $P = \{0, 6\}$ ,  $R_1 = \mathbf{I}_0(R_1)$ ,  $R_2 = \mathbf{I}_6(R_2)$ , and  $S_x = P \cup T_x(R_1 \cup R_2)$ . Then  $S_x$  and  $S_{-x}$  are  $Z$ -related.

By varying the choices of  $R_1$ ,  $R_2$ , and  $x$ , case 4 generates 12 of the 19  $Z$ -related pairs of 6 or fewer notes in 12-tet. (See Table 1.) There are a total of 85  $Z$ -related pairs in 8-tet through 14-tet, not including those pairs where the cardinality is greater than half of the number of pcs. Case 2 generates 50 of these pairs, or 59%. (Of course one of the remaining pairs corresponds to case 3.)

$R_1$	$R_2$	$x$	$S_x$	$S_{-x}$
{0}	{3}	1	{0, 1, 4, 6} $\in$ 4-Z15	{11, 0, 2, 6} $\in$ 4-Z29
{0}	{2, 4}	1	{0, 1, 3, 5, 6} $\in$ 5-Z12	{11, 0, 1, 3, 6} $\in$ 5-Z36
{-2, 2}	{3}	1	{11, 0, 3, 4, 6} $\in$ 5-Z18	{9, 0, 1, 2, 6} $\in$ 5-Z38
{-1.5, 1.5}	{2.5, 3.5}	0.5	{10, 0, 1, 2, 3, 6} $\in$ 6-Z39	{11, 0, 2, 3, 4, 6} $\in$ 6-Z10
{-2.5, 2.5}	{1.5, 4.5}	0.5	{9, 0, 1, 2, 4, 6} $\in$ 6-Z46	{10, 0, 2, 3, 5, 6} $\in$ 6-Z24
{-2.5, 2.5}	{0.5, 5.5}	1.5	{8, 11, 0, 1, 4, 6} $\in$ 6-Z48	{11, 0, 2, 4, 6, 7} $\in$ 6-Z26
{-1, 0, 1}	{3}	2	{0, 1, 2, 3, 5, 6} $\in$ 6-Z3	{9, 10, 11, 0, 1, 6} $\in$ 6-Z36
{-0.5, 0.5}	{2.5, 3.5}	1.5	{0, 1, 2, 4, 5, 6} $\in$ 6-Z4	{10, 11, 0, 1, 2, 6} $\in$ 6-Z37
{-2, 0, 2}	{3}	1	{9, 11, 0, 1, 2, 6} $\in$ 6-Z40	{11, 0, 1, 3, 4, 6} $\in$ 6-Z11
{-1.5, 1.5}	{2.5, 3.5}	0.5	{11, 0, 2, 3, 4, 6} $\in$ 6-Z10	{10, 0, 1, 2, 3, 6} $\in$ 6-Z39
{-4, 0, 4}	{3}	1	{9, 0, 1, 4, 5, 6} $\in$ 6-Z44	{7, 11, 0, 2, 3, 6} $\in$ 6-Z19
{-1, 1, 6}	{3}	2	{0, 1, 3, 5, 6} $\in$ 6-Z25	{9, 11, 0, 1, 4, 6} $\in$ 6-Z47

Table 1: The  $Z$ -related pcsets  $S_x$  and  $S_{-x}$  in 12-tet, where  $S_x = \{0, 6\} \cup \mathbf{T}_x(R_1 \cup R_2)$ .

Case 3 can be realized using Soderberg's  $Q$ -inversion, but to do so one must understand the basic structure of these sets. In other words, one must understand Case 3 to find the appropriate  $Q$ -inversion.

**Case 4a** (Bullough) We can extend Case 1 in the following manner. Let  $P$  be the  $m$ -fold division of the octave that contains  $\{0\}$ ,  $P = \left\{ \frac{12j}{m} \right\}_{j=0}^{m-1}$ . Divide the smallest interval of  $P$  into  $n$  parts,  $Q = \left\{ \frac{12k}{mn} \right\}_{k=0}^{n-1}$ , and transpose this by an arbitrary interval,  $\mathbf{T}_x(Q)$ . Then

$$P \cup \mathbf{T}_x(Q) Z P \cup \mathbf{T}_{x+\frac{12k}{mn}}(Q).$$

**Case 4b** (Bullough) This is the same as Case 5a with the exception that members of  $Q$  are not necessarily equal to  $\frac{12k}{mn}$ . Rather, each member of  $Q$ ,  $q_k$ , belongs to the  $m$ -fold division of the octave that contains  $\frac{12k}{mn}$ . That is,  $Q = \{q_k\}_{k=0}^{n-1}$ , where  $q_k \equiv \frac{12k}{mn} \pmod{\frac{12}{m}}$ . With this modification, the result in Case 5a also obtains.

## 4 The Z-relation as an algebraic property

Rosenblatt and others represent sets in  $\mathbb{R}$  or  $\mathbb{R}/\mathbb{Z}$  as polynomials. A multiset  $\mathcal{D}$  is represented by a finite distribution  $\sum_{d \in \mathcal{D}} a_d \delta_d$ , where  $a_d \in \mathbb{Z}$  is the number of times  $d$  appears in the multiset, and a polynomial  $D(x) = \sum_{d \in \mathcal{D}} a_d x^d$ .

A spectral unit is a function in  $K[G]$  such that  $U * U^* = \delta_0$ . (For our purposes, we can assume that  $K = \mathbb{C}$  and  $G$  is either  $\mathbb{R}/\mathbb{Z}$  or  $\mathbb{R}$ ). As a polynomial  $U(x)U^*(x) = 1$ .

**Theorem (Rosenblatt)** *If  $D, E \in K[G]$ , then  $D$  and  $E$  are homometric if and only if there exists a spectral unit  $U \in K[G]$  such that  $U * D = E$ .*

Alternatively,  $D$  and  $E$  are homometric if and only if  $U(x)D(x) = E(x)$ . Thus, the existence of homometric sets relates to the factorization of polynomials making the Z-relation is an algebraic property. Whether this algebraic property has a relevant musical interpretation depends on the particular case. For example, the existence of the exceptional four-note homometric pair in 13-tet would seem to be more related to factorization of polynomials than to musical principles.

## 5 Bibliography

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